Characterisation of the Degree of Musical Non-Markovianity

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Abstract

As an aid for musical analysis, in computational musicology mathematical and informatics tools have been developed to characterise quantitatively some aspects of musical compositions. A musical composition can be attributed by ear a certain amount of memory. These results are associated with repetitions and similarities of the patterns in musical scores. To higher variations, a lower amount of memory is perceived. However, the musical memory of a score has never been quantitatively defined. Here we aim to give such a measure following an approach similar to that used in physics to quantify the memory (non-Markovianity) of open quantum systems. We apply this measure to some existing musical compositions, showing that the results obtained via this quantifier agree with what one expects by ear. The musical non-Markovianity quantifier can thus be used as a new tool that can aid quantitative musical analysis. It can also lead to future quantum computing controllers to manipulate structures in the framework of generative music.

Keywords: memory, non-Markovianity, open quantum systems, computational musicology, pattern repetition.

1 Introduction

1.1 Mathematics and Music

Cross-fertilisation between music and science, in particular mathematics and computer science, nowadays constitutes a flourishing research field, involving both the creation of new musical pieces and the analysis of existing ones. The main focus of this research is to develop a physics-inspired strategy to investigate the amount of thematic memory in a musical piece. This article presents a heuristic approach to measure musical memory.

Applications of mathematics and computer science are particularly relevant for computational musicology. In fact, in computational musicology, methods to quantify specific aspects of musical compositions mathematically have been developed (Jedrzejewski, 2019a; Montiel & Peck, 2018; Bel & Vecchione, 1993). They are used both as an aid for musical analysis, and to investigate some aspects of samples of compositions (Mazzola, 2007; Mazzola, 2002). In particular, these methods have been applied to analyse: musical theory in a geometrical way (M. et al., 2006); gesture in music (Mazzola & Andreatta, 2007; Mannone, 2018a; Arias, 2018); counterpoint; interpretation; harmony (Mazzola, 2002); common patterns in sound amplitude (Mendes et al., 2011); chord-sequences and voice-leading in the light of topology (Tymoczko, 2006; Tymoczko, 2020); and symmetries between chords in lattices called *Tonnetze* (Amiot, 2017; Jedrzejewski, 2019b). Formal tools also include using the abstract power of category theory in order to formalise musical structures, including relationships within music theory and between music theory and musical performance (Popoff et al., 2019; Mannone, 2018b; Mazzola, 2002; Mannone, 2018a; Arias, 2018; Jedrzejewski, 2019a).

Techniques of information retrieval, often applied to music, can be connected with quantum mechanics (Piwowarski et al., 2010; Di Buccio et al., 2009), but, to the best of our knowledge, quantum mechanics has not yet been applied to music information retrieval. Thus, we propose a quantum-based approach to musical information retrieval.

1.2 Quantum Mechanics and Music

Among mathematical tools, the formalism of quantum mechanics is the object of growing attention (Miranda, 2021; beim Graben & Blutner, 2020; beim Graben & Mannone, 2020; Rocchesso & Mannone, 2020; Fugiel, 2022; Putz & Svozil, 2017).

In particular, the physicists beim Graben and Blutner gave mathematical substance to conceptual analogies between musical tonal rules and force fields, exploiting the definitions of gauge fields (beim Graben & Blutner, 2020). As a development of this research, it has been proven that pitch quantisation (musical notes as sets of quantised pitches, connected through chords) emerges from symmetry-breaking and from the eigenvalues of the Schödinger equation (beim Graben & Mannone, 2020).

The pioneering contribution of Gabor (1947) exploits the quantum paradigm to analyse sounds and to formalise the idea of *quantum of sound*. Starting from Gabor's hints, and considering the use of the human voice as a probe to investigate the world of sound,[1] the quantum mechanical formalism has been used to formalise the basics of the human voice (phonation, turbulence, and myoelasticity), allowing some first experiments in sound analysis and synthesis (Rocchesso & Mannone, 2020).

The interplay between musical and extra-musical ideas has been contextualized within quantum semantics (Dalla Chiara et al., 2015). Also, the Dirac notation used in quantum mechanics, jointly with the basic ideas of destructive quantum measure, has been applied to analyse visual structures and translate them into musical structures (Mannone et al., 2020).

The quantum paradigm has been exploited for interactive music-production interfaces, such as *Qubits* (Kulpa et al., 2020). In this framework, users can select sound trajectories between maximum noise/least pitch and maximum pitch/least noise (Kulpa et al., 2020). Noticeably, quantum computing has been applied to music (Miranda, 2021). There have been other proposals of sonifications based on quantum mechanics ideas and techniques (Kontogeorgakopoulos & Bugarth, 2014), including sonification of dynamic behaviours (Cádiz & Ramos, 2014). Time series appear to be relevant in the framework of generative music (Dean, 2017), and they might be related to the idea of *memory* (Toop, 2004).

Some of the applications mentioned concern mathematical techniques for quantum systems that demonstrate effectiveness in analysing music (Dalla Chiara et al., 2015; Mannone et al., 2020; Kontogeorgakopoulos & Bugarth, 2014; Dean, 2017). Some other studies involve the direct application of the basics of quantum systems (Gabor, 1947; Rocchesso & Mannone, 2020) and quantum computing in sound and music (Miranda, 2021; Cádiz & Ramos, 2014; beim Graben & Blutner, 2020; beim Graben & Mannone, 2020; Kulpa et al., 2020). While there is a flourishing of studies on quantum computing applied to music, here we focus in particular on methods and concepts from open quantum systems. Quantum computing is a branch of computer science, while quantum mechanics is a branch of physics. They both start from the quantum definition of states and measurements.

1.3 Memory, Music and Quantum Mechanics

In this article, we develop an application of quantum mechanics to investigate the amount of "memory" in musical scores. Memory, understood here as the repetition and recognisability of motivic, rhythmic and harmonic elements, is a key element in music composition, analysis, and even intuitive understanding for non-expert musical listeners. Because memory is strictly related to degrees of recognisability and techniques of musical development, it constitutes an essential element of generative music. We may, in fact, envisage memory-based codes being used to generate music according to the *degree of similarity* to some model that we would like to attain. We may want to have the basic

[1] Vocal imitations capture essential features of sounds; see (Lemaitre & Rocchesso, 2014).

idea of a generative musical piece to be "recognisable". In this framework, the following approach might help.

Much music characteristically presents repetitions of segments.[2] The idea of "repetition" in music is complex. There can be repetitions of entire segments, variations, and thematic cells appearing in different parts of an orchestral score. For the sake of simplicity, we focus here on short and simple fragments, distinguishing between maximal memory in the case of two identical segments, and minimal memory when no similarity between the two segments can be envisaged.

The amount of repetition can in some way be considered to be related to the degree of memory of the composition (Cherlin, 1998; Mazzola et al., 2011). There are algorithms to find the most repeated segments (Meredith et al., 2001; Meredith et al., 2002; Utgoff & Kirlin, 2006), to compose music (Bresson, 2006; Collins et al., 2011; Collins, 2011), to orchestrate (Mannone, 2013), and to improvise in real time (Assayag et al., 2006; Cont et al., 2010; Noll et al., 2006). Repetition in music is important, and algorithms have also been used, either to generate music or to detect repeated segments, in order to characterise segments as something definite (Schenker, 1954; Temperley, 2001; Lerdahl & Jackendoff, 1983; Sciarrino, 1998; Reti, 1978; Kramer, 1992). In particular, Markov chains and hidden Markov chains (Rabiner & Juang, 1986; Flexer et al., 2005; Amiot et al., 2006) have been used in the techniques of score following (Assayag et al., 2006; Cont et al., 2010), analysing chord sequences in jazz improvisation (Franz, 1998), studying spectral similarity (Flexer et al., 2005), and composing Markovian stochastic music (Xenakis, 1963).

Memory is also an important concept in the physics of open systems, that is systems embedded in an environment. It is associated with the concept of non-Markovianity in the dynamics of the system, as distinct from the concept of Markovianity that characterises a memoryless evolution (Breuer & Petruccione, 2002). Therefore, we shall use the term non-Markovian music to characterise the presence of memory in a musical score. While the quantification of the degree of memory in open systems is still an open problem, various criteria have recently been developed in order to quantify it (Andersson et al., 2010; Breuer et al., 2009; Rivas et al., 2010). Each criterion is based on a particular computational aspect. In general, criteria aim to quantify the passage from a Markovian dynamics to a non-Markovian dynamics. There are non-Markovian dynamics when correlations between the system and the environment occur as a product of a joint evolution (Lo Gullo et al., 2014). The relationships between various criteria have recently started to be studied in detail (Mannone et al., 2013).

1.4 Our Contribution

Here we shall address the question: is it possible to quantify the degree of memory in existing compositions? In fact, while algorithms to find a theme in a piece of music have been developed, quantification of the degree of memory in a composition, to the best of our knowledge, has never been addressed.

The aim of this paper is to develop a method to define and measure the Musical non-Markovianity degree of musical compositions, adopting concepts and mathematical techniques used in quantum physics. Among the various criteria available to measure the degree of memory, the one that will be applied to the case of musical composition is that which utilises the distance between matrices (Breuer et al., 2009).

In order to define quantitatively the degree of memory in a musical composition, we follow the way in which the degree of memory is utilised, in the context of the theory of open quantum systems (Cover & Thomas, 2006; Preskill, 2004; Breuer & Petruccione, 2002). It is, however, important to note that the conception of memory, and thus the definition of non-Markovianity, conceived expressly for music, must be different to the definition considered in the quantum case, where comparisons are made between different states at the *same* instant. In a musical composition, we compare segments of finite duration starting at *different* time instants of the same musical piece. The common aspect is the idea of memory as conservation of characteristics that make a pattern distinguishable from another, like distinguishing one state from another.

[2] We assume here notated Western music. However, our technique can be applied to every musical genre, provided that a MIDI rendition can be obtained.

As an application, the degree of memory of three different compositions will be quantified – these already by ear appear to present a very different degree of memory – in order to see if the results obtained align with listeners' expectations.

The structure of this article is as follows. In Section 2.1 we briefly discuss the use of entropy and then, in Section 2.3, we introduce non-Markovianity criteria used in quantum physics, giving more details about which ones are conceptually applicable to musical cases, and in which way. In Section 3 we give technical information about the musical matrices defined and the algorithm developed to find them, and then we apply our method to fictitious examples. In Section 4, we define musical non-Markovianity. In Section 5, we apply the same method to existing compositions, giving numerical results. In Section 6 we discuss possible automation of the proposed method. In Section 7 we give some conclusions. In the Appendix there is an example of distribution matrices.

2 Mathematical Tools

2.1 Relative Entropy as a Measure of Memory

In music, the repetition of some meaningful unit is an important feature. The unit "theme" can be taken to be such a meaningful unit. It usually appears at the beginning of the composition and has characteristics that are repeated during the piece. When a clear "theme" cannot be recognised, it is still possible to individuate an initial musical idea. The repetitions of recognisable units can be taken as an indication of the "memory" embedded in the composition. To be able to quantify some regularity in the score, one must refer to measurable parameters relative to the score, such as pitches, durations and intensities.

Another way to analyse the regularity of the structure in a musical composition is to make a measure of the randomness. These two aspects – the amount of repetitions of some musical pattern, and the degree of randomness in the entire composition – can be useful to characterise the musical score.

To quantify the amount of randomness, entropy can be used (Cover & Thomas, 2006). Xenakis computed probability transitions and measures of entropy in musical segments, evaluating the amount of randomness in a piece (Di Scipio, 1998; Xenakis, 1963). Xenakis also made a distinction between conditions of growing entropy characterising perturbative states, and stationary states as equilibrium points (Di Scipio, 1998). The entropy $H(X)$ of a string of N elements of a random variable X, with the measurable parameter x, is defined as:

$$H_{P_I}(X) = -\sum_{i=1}^{N} p(x_i) \log_2 p(x_i), \tag{1}$$

where $p(x)$ is the probability that the quantity X has the value x_i for the i$^{\text{th}}$ element of the string, $H(X)$ upper limit being $H_{max} = \log_2 N$, and its lower limit zero. The magnitude of $H(X)$ is taken to quantify the amount of randomness in the musical segment.

Entropy has in fact been used to quantify the amount of randomness of sequences of pitches in a musical score (Xenakis, 1963). In that case, given a string of N notes as elements, the measurable parameter x has been identified with values that are related to the value of the pitch.

To evaluate memory, which is a measure of the repetition of segments in a chosen musical score, we need to take into account the order of pitches. Relative entropy, which takes into account pairwise comparisons between the elements of two strings, is useful for this purpose. One can argue that two musical compositions with similar values of relative entropy may have a similar structure. Relative entropy is obtained by comparing the probability of two different distributions of identical objects. If we measure the amount of randomness of a string through a probability distribution, we can then compare two strings through two probability distributions. Thus, if there are two strings, their relative entropy can be computed through the Kullback-Leibler divergence $D(X)$ (Cover & Thomas, 2006, p. 19; MacKay, 2003), defined as:

$$D(X) = -\sum_{i=1}^{N} p(x_i) \log_2 \frac{p(x_i)}{q(x_i)}, \tag{2}$$

where $p(x)$ is the probability that the quantity X has the value x_i for the i^{th} element of the first string, and where $q(x)$ is the corresponding probability for the second string.

The Kullback-Leibler divergence in its symmetrised form has been used to evaluate the degree of similarity within a musical composition (Cont et al., 2010), giving a measure of the most repeated musical segments.

When applied to musical strings of pitches, the use of $D(x)$ is beset with certain difficulties. First, $D(X)$ is not commutative; second, it is necessary that the two segments contain the same number of symbols. The non-commutativity implies that probabilities do not appear in a symmetric way depending on which order is utilised: $p|q$ is different from $q|p$. For example, the comparison between a musical segment containing the notes C and D and a segment also containing E that uses relative entropy, requires neglecting the E in the second segment. If we eventually consider some rests in place of the missing notes in the shorter string, we obtain a theme that is different from the original one, because rests become a characterising aspect of the theme. We notice that, in this approach, we consider one parameter only; however, musical memory is not only defined by pitch, but also by combinations of pitch, rhythm, loudness, etc. After a short overview of a recent application of entropy to musical analysis (Section 2.2), we will present a different formal tool that will allow us to investigate memory in musical pieces (Section 2.3).

2.2 Entropy and Musical Analysis

Recently, the idea of musical entropy has been exploited to quantify the amount of rhythmic activity, in a study of the definition of musical entropy based on Fourier coefficients (Amiot, 2020). The application of Fourier coefficients to sound analysis is well known. In the framework of mathematical music theory, some authors analysed Fourier coefficients to evaluate distributions of rhythms or pitches, in order to compare quantitatively rhythmic patterns for one, and pitch distributions – scales, motives – for the other (Amiot, 2016). The drawback of this approach is that each parameter, be it rhythm or pitch, is analysed independently, yet we need to consider multiple parameters simultaneously. Recently, the Fourier coefficients for mathematical music theory have been combined with entropy. The entropy of Fourier-coefficient distribution has been theoretically defined (Amiot, 2020) and applied to analyse the rhythm of a few bars of an orchestral piece (Favali, 2020). Entropy is maximum when the rhythm is more homogeneous. This happens with silence. When activity (the sound itself) starts, entropy diminishes. It starts rising again when the rhythm becomes so complex that the perception is once again of homogeneity. Thus, the maximum entropy corresponds to two opposed situations. We can imagine a range from minimal entropy in the case of silence to maximal entropy in the case of rhythmic chaos. This also happens in the visual arts.[3] However, beyond "perceivable" chaos, we have the limit of white noise, and of an extreme rhythmic tessellation, where no distinction between beats is possible. The interpretation of entropy is not trivial, because the same value can correspond to very different musical situations – complete silence versus uninterrupted superposed rhythms – and thus it does not appear to be a good candidate to distinguish musical passages and to assess their degree of musical memory.

In addition, musical memory does not depend upon a single parameter, such as the rhythm just considered. To characterise the score content completely, we need a more complete analysis, one that requires the use of several parameters, such as intensities and durations. In fact, musical memory is more complex than the specific amount of activity occurring in a single parameter. For this reason, and because of the aforementioned ambiguity in musical entropy and the restrictions imposed by the use of relative entropy, we will use other mathematical tools. We will be inspired by techniques to measure memory for open quantum systems, the non-Markovianity criteria, and adapting them to music. We will derive a non-Markovianity degree, a unique quantifier whose result is evident and not subject to interpretation as is entropy. As we will see, in our analysis the amount of memory related to the chosen parameters is evident.

2.3 Non-Markovianity Criteria

This section aims to give a quantitative characterisation of non-Markovianity in music. We build a measure following a similar path as that used in the theory of open quantum systems (OQS).

[3] Relationships between values of entropy and organisation and regularity in visual artworks are discussed in (Burns, 2015).

Account must nevertheless be taken that memory in music can be different from that which is used in the physics of OQS. In the latter, loss of memory occurs when two states (represented by matrices), initially distinct, become progressively indistinguishable; i.e., the lower the distinguishability, the lower the memory. In a musical score, by contrast, memory is associated with repetitions of patterns. To a higher number of repetitions corresponds a higher amount of memory, i.e., the lower the distinguishability among segments, the higher the memory.

Quantification of non-Markovianity in the theory of OQS have been addressed recently by (Breuer & Petruccione, 2002), introducing different criteria with appropriated quantifiers. A given quantifier may be simpler in application and understanding than another. Among the quantifiers used in OQS, one (Breuer et al., 2009) quantifies memory by considering the variation of the distances between two quantum states with time. The increase or decrease of this quantifier is associated with persistence or loss of memory.

Other criteria study the density matrix that describes the state, the master equation (Breuer & Petruccione, 2002). One criterion looks at the separability of the map: when it is separable, the dynamics are Markovian; otherwise, the dynamics are non-Markovian (Rivas et al., 2010). One other criterion looks at the signs of coefficients in the master equation (Andersson et al., 2010). In general, the different criteria of non-Markovianity do not always agree. For example, to describe the presence of memory revivals in different time regions (Chruscinski & Kossakowski, 2010), the comparison between them is an open problem (Mannone et al., 2013).

We choose to quantify the presence of memory in musical scores using a criterion that makes direct use of matrices. This is because, for a musical score, we do not have anything corresponding to a map or a master equation, yet it is instead possible to represent a musical segment as a matrix. The distance between two states is given by the trace distance of the matrices ρ_1 and ρ_2 representing the states (Breuer & Petruccione, 2002, p. 105; Breuer et al., 2009):

$$D(\rho_1, \rho_2) = \frac{1}{2} tr|\rho_1 - \rho_2|, \tag{3}$$

where, if A is a Hermitian matrix, $|A| = \sqrt{A^\dagger A}$.[4] The rate of variation of the distance D is defined as

$$\sigma = \sigma(\rho_1, \rho_2) = \frac{d}{dt} D(\rho_1, \rho_2). \tag{4}$$

When $\sigma > 0$, the distance is increasing, and thus the distinguishability between states is preserved. When $\sigma \leq 0$, the distance is decreasing, and the memory is progressively lost. This is a crucial point, and it is the core of the considered non-Markovianity criterion from physics. If, at time 0, we have two different states, and if their difference increases over time, then we say that we keep the *memory of their distinction*.[5] However, if we start with two different states and, across time evolution, they evolve to become similar, and even indistinguishable, their difference is lost: that is, the memory of their separation is lost. This is why we state $\sigma > 0 \rightarrow$ memory preserved, $\sigma \leq 0 \rightarrow$ memory lost.

3 Musical Matrices

In the physics of OQS, matrices ρ_1, ρ_2 represent quantum states; here, matrices must be related to the content of a musical composition. To adopt similar methods in musical compositions to quantify the amount of frequency, we shall build matrices starting from a musical score. In order to construct these matrices, we shall convert into numbers the symbolic information contained in a musical score.

A musical composition is divisible into segments, each segment corresponding with a set of numerical parameters relative to frequencies, intensities, times of start and durations. Matrices can be constructed in terms of these numerical values. The matrices we use represent the distribution relative to every note in each segment around the mean value of these parameters.

[4] Hermitian matrices are symmetric with respect to the exchange of indices of their elements. Our matrices are special cases of Hermitian matrices, keeping their symmetry, and imaginary parts equal to zero. Similarly, real numbers can be seen as complex numbers whose imaginary part is zero.

[5] In our musical framework, the "state at time zero" would be the initial musical segment (of finite time duration).

For the pitch of notes, we use differences in semitones with respect to middle C ($0 = C4$, $-1 = B3$, $1 = C\sharp 4$). For intensities, we use dimensionless numbers to indicate relative intensity indications in musical scores. In particular, we choose 90 for *ffff*, 80 for *fff*, 70 for *ff*, 60 for *f*, 50 for *mf*, 40 for *mp*, 30 for *p*, and so on. For time, we have used dimensionless units: that is, the ratio between the duration of the examined note and the metronome mark.[6] In particular, we have taken 1 for semiquavers, 2 for quavers, and so on. The duration of a rest will be counted as the time before the start of the following note. The couples of parameters that we have chosen to characterise our matrices are: duration-intensity, frequency-start, start-intensity, frequency-duration, and frequency-intensity.

For each musical segment, we have first obtained the mean value relative to each parameter. Then, we have evaluated the distance (normalised between 0 and 1) of each parameter of a note in the segment with respect to its mean values. The range of distances from the first parameter and the range of distance from the second has been divided into equal parts.

Having established a segment, let us consider a note and a couple of its parameters, for example, duration and intensity. We evaluate the difference between the value of each parameter of the note and the corresponding mean value in the segment. We then discretise the bidimensional domain of values, defining a reticular step. Given a note and its parameters, we see if the normalised difference is contained in each element of the lattice. The number of elements in the lattice represents the musical matrix.

For each couple of parameters, one constructs a matrix relative to a numerical segment. It is clear that it is possible to divide the distance range 0-1 into several parts. In our analysis, we shall limit ourselves to matrices with four rows and four columns. An example of this procedure in the case of the frequency-start matrix, and corresponding to a given musical segment, is described in the Appendix. This procedure can be applied through the use of an algorithm that we have developed.

In the Appendix, in order to provide a worked example, we chose a minimal segment, irrespective of its time-division, to build a 4 x 4 matrix. The use of 4 x 4 matrices is a choice. We have proved that with 4 x 4 matrices we already acquire information on musical memory. The structure of matrices depends upon the parameters we choose to define the system, the space characteristics, and the degree of chosen fine-graining. In fact, we built up 4 x 4 matrices to analyse the memory of pairs of variables. A four-part matrix is precise enough to capture quantitative information corresponding to our qualitative assessments. The entire method could be made more precise, for example, with 9x9 matrices, using multiple parameters and allowing a fine-grained analysis of musical passages. However, the theoretical discussion and motivation would remain the same.

4 Musical Non-Markovianity

In the following, we shall analyse the structure of a generic musical composition that will be used as the basis to define musical memory.

Most musical compositions are divisible into sections. The structure of the entire musical piece is determined by the structure of sections. The most common form for a single movement in classical compositions is arguably A-B-A'. Let us consider a simplified structure A-B. Both sections may contain several segments; let us suppose A_1, A_2, A_3 for Section A, and B_1, B_2, B_3 for Section B. The natural succession in time of the segments in a score is thus the following:

$$\mathrm{A}_1\ \mathrm{A}_2\ \mathrm{A}_3\ \mathrm{B}_1\ \mathrm{B}_2\ \mathrm{B}_3 \tag{5}$$

In order to define a non-Markovianity criterion, we reorder these segments in time to compare them. Sections A and B are ordered as:

$$\begin{matrix} A & B \\ A & B \\ A & B \end{matrix} \tag{6}$$

[6] The metronome mark is an indication at the beginning of a musical score indicating the correspondence between the predominant rhythmic value of the piece (for example, a crotchet or a quaver) and the number of beats per minute (bpm).

$$t\Big\downarrow \begin{array}{cc} A_1 & B_1 \\ A_2 & B_2 \\ A_3 & B_3 \end{array} \tag{7}$$

with time flowing downward.

To each segment, we assign a set of two-dimensional matrices, each matrix representing the distribution of a couple of variables for each note (for example, frequency and duration). The exact procedure to build these matrices has been outlined in Section 3. Here we proceed to apply the criterion of non-Markovianity described in Section 2.3 to the above canonical structure, constructing a matrix that corresponds to a musical segment.

To the musical composition above correspond the set of matrices:

$$t\Big\downarrow \begin{array}{cc} \rho_1^A & \rho_1^B \\ \rho_2^A & \rho_2^B \\ \rho_3^A & \rho_3^B \end{array} \tag{8}$$

The trace distance defined in Eq. 3 between simultaneous segments ρ_i^A, ρ_i^B is $D_i = D_i(\rho_i^A, \rho_i^B)$. It measures the distance between the segments ρ_i^A and ρ_i^B. The variation rate σ in the case of music can be defined as $\frac{\Delta D}{\Delta t}$ (we are considering finite time intervals), where $\Delta D_{i+1} = D_{i+1}(\rho_{i+1}^A, \rho_{i+1}^B) - D_i(\rho_i^A, \rho_i^B)$ and $\Delta t_{i+1} = t_{start,\, i+1} - t_{start,\, i}$.

Because of the normalisation of the number of notes, all time-intervals Δt become equal, and we simply define the rates as $\sigma_i = D_{i+1} - D_i$. We consider only positive values, which represent the case of increasing distance.

Contrary to the case of physical systems, where, if the distance decreases memory is lost, in a musical composition memory is preserved if the distance is constant or decreases. In fact, the increasing distance mentioned above corresponds to the increasing σ of the non-Markovianity criterion. Musical non-Markovianity is defined as $1 - \sigma$ and thus, if σ increases, $1 - \sigma$ decreases. In physics, the lower the distinguishability, the lower the memory; while in music, the lower the distinguishability, the higher the memory. In fact, to successive equal musical segments, which means a maximum musical memory, there correspond equal matrices.

Now we define a quantitative measure of the amount of memory, or of non-Markovianity, in a musical composition. For open quantum systems, the degree $\mathcal{N}_{max}$ of non-Markovianity is defined as an integration of $\sigma(\rho_1, \rho_2)$ in Eq. (4) (Breuer et al., 2009), over time. The contribution to the integral is taken only over regions with $\sigma(\rho_1, \rho_2) > 0$, and maximisation is then performed over all the possible initial states. In the musical case, however, we want to define the memory degree for each individual musical piece and thus for a single initial state, not for all possible initial states.

We can partially preserve the idea of integration used in physics. Since in a generic musical composition there is not a statistical structure, we can utilise a discretised version of $\mathcal{N}_{max}$, with the sum of positive rates σ_i instead of the integral, where no maximisation is performed: we call it $n = \sum_{i(if\; \sigma_i > 0)} \sigma_i$.[7] Then, we renormalise this quantity in order to define an $\mathcal{N}$ comprised between 0 and 1: $\mathcal{N} = \frac{\sum_{i(if\; \sigma_i > 0)} \sigma_i}{1 + \sum_{i(if\; \sigma_i > 0)} \sigma_i}$, with the form $\frac{n}{1+n}$ (Rivas et al., 2010). Thus, $0 \leq \mathcal{N} \leq 1$.

Finally, we define our quantifier of musical non-Markovianity as $\mathcal{M} = 1 - \mathcal{N}$, to characterise the amount of memory in musical compositions. The quantifier $\mathcal{M}$ is equal to 1 when the thematic memory is at the maximum, i.e., when in a musical composition there are only repetitions of the same segment. To lower $\mathcal{M}$, there corresponds a lower amount of musical memory. In a nutshell, musical non-Markovianity is defined as $1 - \sigma$, where σ is the quantifier defined for the non-Markovianity criterion considered. Thus, if σ increases, $1 - \sigma$ decreases. This is why we are considering positive values of σ.

The quantifier $\mathcal{M}$ is then:

[7] Regarding the normalisation on the number of notes, we are considering the number of notes as the pulses, the musical events. While building up a new model, it is necessary to make some choices, which can be refined in further research.

$$\mathcal{M} = 1 - \frac{\sum_{i\,(if\,\sigma_i>0)} \sigma_i}{1 + \sum_{i\,(if\,\sigma_i>0)} \sigma_i} = \frac{1}{1 + \sum_{i\,(if\,\sigma_i>0)} \sigma_i}. \tag{9}$$

$\mathcal{M}$ as defined in Eq. 9 does not differentiate between cases with an identical sum of positive rates $\sum_{i,\,\sigma_i>0} \sigma_i$, because it does not give any information about the total number of rates, positive, negative and null. For example, let us consider a score with ten total rates, with only two positive ones, and with sum $\sum_{i,\,\sigma_i>0} \sigma_i^*$: the positive contribution has the proportion of 2 over 10. (With σ^*, we indicate a specific evaluation of σ.) By contrast, let us consider another composition, with only three total rates, with two positive ones, and with identical sum $\sum_{i,\,\sigma_i>0} \sigma_i^*$: the positive contribution in this case is 2 over 3. The information given by Eq. 9 is the same for the two compositions, and does not take into account the different proportions (2/10 *versus* 2/3).

This situation can be easily solved by introducing a correction factor, $r = \frac{n_+}{n_T}$, defined as the ratio between the number of positive rates n_+ and the number of total rates n_T. So we define a new quantifier:

$$\mathcal{M}_C = \frac{1}{1 + r \sum_{i\,(if\,\sigma_i>0)} \sigma_i}. \tag{10}$$

$\mathcal{M}_C$ appears to be a better measure of the memory in musical structures than $\mathcal{M}$.

Now we give two simple musical examples. The first shows two identical bars (Fig. 1), and the second presents two very different bars (Fig. 2). The expectation is for the maximum value of memory for the first example, and for a very low value for the second one.

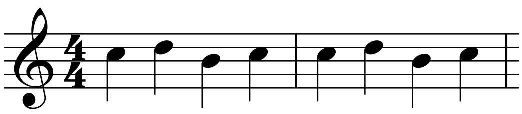

Figure 1: In the case of two identical bars, the amount of memory is maximal.

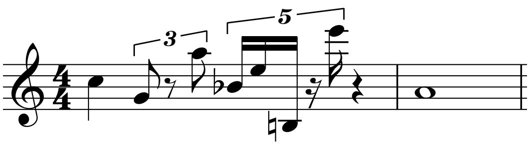

Figure 2: The first bar is a quasi-random sequence of pitches and durations, while the second is totally different: in this case the memory for the parameters frequency-start is 0.3, a very low value, since $0 \leq \mathcal{M}_C \leq 1$. Neglecting onsets, we would obtain a similar degree of memory for a segment with the same pitches in a different order. However, to the ear of a listener, a rather chaotic segment as that in the first bar might lead to a very similar perception irrespectively of the order of pitches.

Let us now consider the parameters frequency and start, and let us evaluate the amount of memory using $\mathcal{M}_C$. In Fig. 1, with its two identical bars, the trace distance (defined in Eq. 3) between the matrices for the first and second bars is zero, and then the amount of memory is immediately maximal, i.e., $\mathcal{M}_C = 1$. However, in Fig. 2, we have two completely uncorrelated bars: the first bar contains quasi-random pitches and duration-start, while the second bar has only a semibreve. In this case, the amount of memory $\mathcal{M}_C$ is equal to 0.3, a low value since $0 \leq \mathcal{M}_C \leq 1$. We have utilised $\mathcal{M}_C$, although in these two examples there are only two bars, and there is only one rate σ. Thus, in this particular case, the results given by $\mathcal{M}$ and $\mathcal{M}_C$ are equivalent.

5 Application of a Non-Markovianity Criterion

We now apply the methods of the previous sections, and the technique described in Section 3, to some existing compositions.

The chosen compositions are the vocal part of the aria *Dolente Immagine* that is the eighth piece from *Quindici Composizioni da Camera* by Vincenzo Bellini (Bellini, 1935); the first piece from the oboe suite *Solo* by Bruno Maderna (Maderna, 1971); and the third of *Metamorphosis I–V* by Philip Glass (Glass, 1988). They are therefore examples of an Italian romantic vocal composition, an avant-garde Italian composition, and a contemporary minimalist work, respectively. These compositions have been chosen because they appear to have, on first hearing, rather different degrees of musical memory.

The compositions will be analysed using both the musical non-Markovianity degree $\mathcal{M}$ given in Eq. 9, and $\mathcal{M}_C$ of Eq. 10. We will see that the use of $\mathcal{M}_C$ allows for a better characterisation of memory in some of the treated cases.

An important step in the analysis is the division of the structure into segments. We will construct tridimensional graphs to represent the structure of pitches, onsets and intensities of musical compositions. The use of tridimensional graphs to study characteristics of sounds has been proposed by Xenakis (1963), and their use to study graphically the orchestration of musical scores has been proposed by Betta (Mannone, 2011). Here, we will briefly examine the structure of the chosen compositions in order to find the optimal subdivision into segments.

Bellini. The score has a structure of type A-B-A', and can be divided into seven periods: Section A has three periods, Section B only one, and Section A' has three. In this research, we divide musical pieces into segments by hand. This step can, however, be automated. The motivation for the subdivision into three sections is due to the tonality change in Period four, and the reprise of the theme and its tonality in Period five. The subdivision into segments is, as discussed, due to reasons of the musical analysis of a classic model. There are some identical parts between Sections A and A', and this fact induces us to expect a high degree of memory. A matrix is then associated (for each couple of parameters, as discussed in Section 3), with each segment. In the case of the Bellini piece, a segment naturally corresponds to a period. Fig. 3 graphically represents pitches, onsets, and intensities of the score, showing some repeated patterns. The intensity in the score is constant, and thus the complete development of the composition can be represented in the time-frequency plane (Fig. 4).

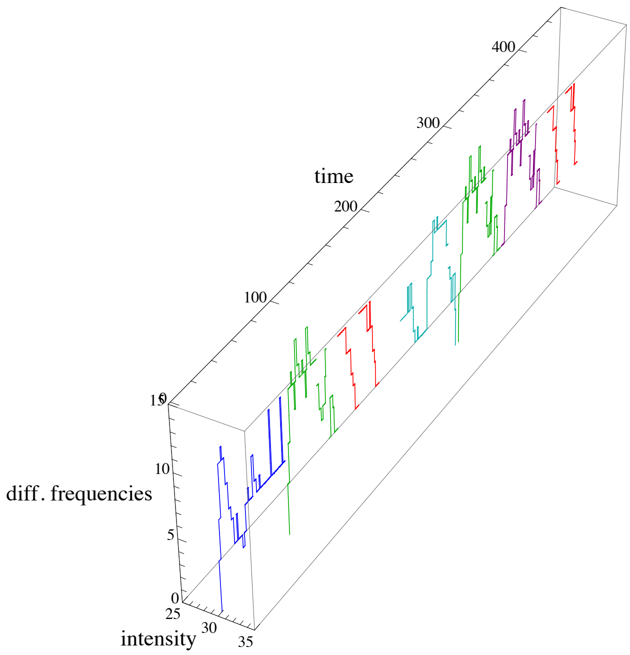

Figure 3: The tridimensional graph represents the vocal part of *Dolente Immagine* by Bellini. The image is flat since the intensity is constant. There are seven periods, and the analysed segments correspond to periods. Periods with identical segments are shown in the same colour. The mean value of $\mathcal{M}_C$ is 0.9.

Maderna. It is evident from the score of the first piece of the oboe suite *Solo*, and from looking at the tridimensional representations of the score in Fig. 5, that the structure is very different from Bellini's composition. In fact, the quantity of repetitions is clearly lower, there are no thematic or

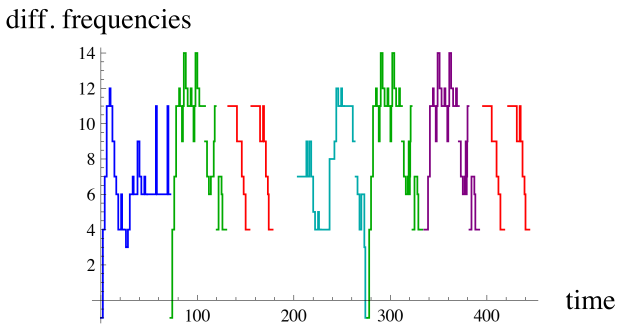

Figure 4: The projection of the graph of Fig. 3 in the time-frequency plane. The intensity in the score is constantly equal to *piano* (30 in our scale), and then the plane time-frequency contains the entire vocal score. The presence of repeated patterns is evident.

other relevant patterns, and there are only a few thematic fragments. So in this case one expects that, with respect to the Bellini composition, there is a lower degree of memory. Since there are few thematic or tonal motivations here, unlike Bellini's work, it is impossible in this case to talk about periods, but only of segments separated by the *respiri e legature* (breaths and slurs), the most natural criterion in this case. We have on this basis divided the score into eleven segments.

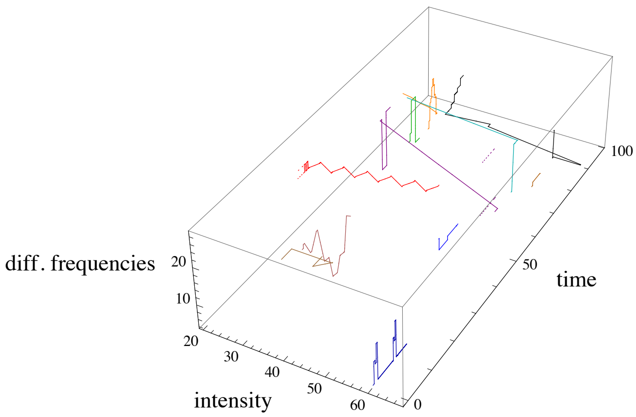

Figure 5: Tridimensional representation of the first piece of the suite *Solo* by Bruno Maderna. The limited quantity of repeated patterns is evident. Each colour identifies a different segment. The mean value of $\mathcal{M}_C$ is 0.6.

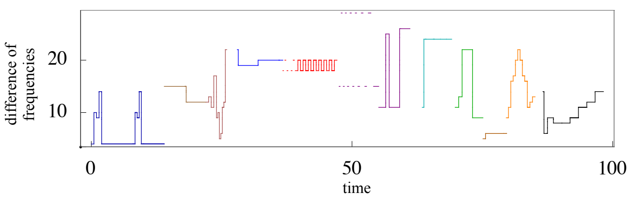

Figure 6: Projection in the time-frequency plane of the graph of Fig. 5.

Glass. The piano composition *Metamorphosis III* is an example of the minimalist style, where there are repetitions with a very small number of pattern changes and variations. The expected degree of memory is, therefore, higher with respect to Maderna's case. We have chosen this example because it is one where it is possible to verify the usefulness of the corrective factor r (Eq. 10). Due to its regularity, the composition has been divided into twenty-two segments, each segment containing four bars (bar 1 of the refrain has not been considered). The two-dimensional graph of Fig. 8 clearly shows the repeated patterns; the tridimensional graph of Fig. 7 also shows the variations of intensity.

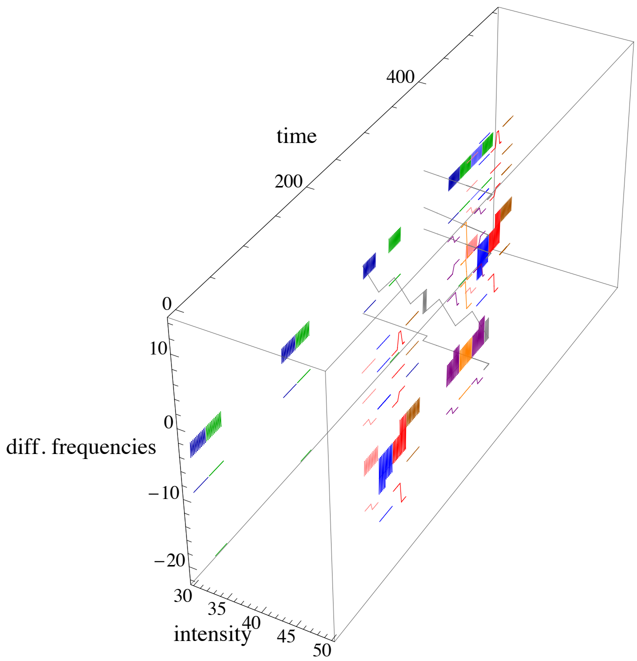

Figure 7: Tridimensional representation of *Metamorphosis III* by Philip Glass. In this graph, we chose the value 1 for the quaver (in the other pieces, 2 was chosen for the quaver). The calculations of the degree of non-Markovianity are unaffected by these variations, because the important elements are the distributions towards the mean values of pitches and onsets. The mean value of $\mathcal{M}_C$ is 0.8.

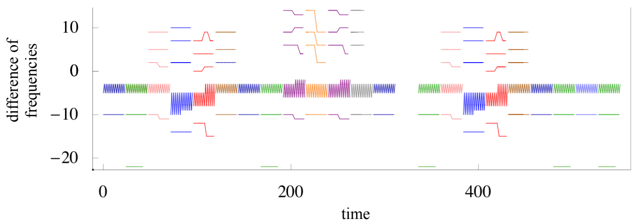

Figure 8: Projection into the time-frequency plane of the graph of Fig. 7.

In the following, we obtain the musical matrices corresponding to the above three musical scores and apply them to analyse the respective memory degree $\mathcal{M}$ and $\mathcal{M}_C$, as defined in Eq. 9 and Eq. 10. The results are given in Table 1.

The values of $\mathcal{M}$ do not appear to correspond with the empirical expectations for all pieces, although it corresponds for the first two pieces. The values of $\mathcal{M}_C$ appear to correspond: in fact, Bellini's and Glass's compositions have an average degree of memory higher than Maderna's, and that of Bellini's composition is higher than Glass's (using the corrective factor). In particular, the mean values of $\mathcal{M}_C$ over the couples of parameters considered are 0.9 for Bellini, 0.6 for Maderna, and 0.8 for Glass. Therefore, we have seen that the degree of memory $\mathcal{M}_C$ is better than $\mathcal{M}$, since it quantifies correctly the memory of Glass's score. In some cases, the values of $\mathcal{M}$ and $\mathcal{M}_C$ are not significantly different, but in others, and depending on the length of the composition, the correction appears decisive. Therefore, it seems that the method can be extended to analyse the degree of memory of musical compositions using a larger set of data.

To apply these methods to analyse musical scores it would be advantageous to automate it in order to obtain the reduction into numbers of the musical parameters contained in the score and the subdivision into segments (Mannone, 2013).

frequency - start

	B	M	G
$\mathcal{M}$	0.97	0.73	0.55
$\mathcal{M}_C$	0.99	0.77	0.97

duration - intensity

	B	M	G
$\mathcal{M}$	0.75	0.40	0.38
$\mathcal{M}_C$	0.82	0.53	0.78

start - intensity

	B	M	G
$\mathcal{M}$	0.86	0.54	0.51
$\mathcal{M}_C$	0.81	0.59	0.86

frequency - duration

	B	M	G
$\mathcal{M}$	0.75	0.56	0.55
$\mathcal{M}_C$	0.82	0.62	0.79

frequency - intensity

	B	M	G
$\mathcal{M}$	0.85	0.55	0.40
$\mathcal{M}_C$	0.94	0.67	0.57

Table 1: Values of the musical non-Markovianity quantifiers $\mathcal{M}$ and $\mathcal{M}_C$ calculated for musical matrices of the examples by Bellini (B), Maderna (M) and Glass (G).

6 Automation

We can wonder if, and to what extent, the process discussed above can be automated. At present, some aspects are automatic – the computation of matrices – and some are not – the division of musical scores into segments and the matrices' comparison. The first issue can be solved with the help of time series-based techniques.[8] The second issue can be addressed with the help of a categorical approach to matrices.[9] Each matrix is considered as a "point" and, thus, matrix comparisons can be easily compared with suitable morphisms. In our research, we might define functors to compare the thematic elements of a piece with those of another piece; and the matrix mappings for a piece with those for the other piece. Thus, the comparison of the degree of memory of different musical compositions can correspond to a functorial structure, which retains all information from single segments (and their associated matrices) to matrix comparisons. The degree of musical memory can thus be seen as an *index* to be associated with such a functorial process. This can be the object of further computational research, leading toward the automation of the main steps of the proposed method.

7 Discussions and Conclusions

We have proposed a method to measure the amount of memory in a musical composition. To quantify musical memory we have constructed quantifiers following a non-Markovianity quantifier used in the theory of open quantum systems (Breuer et al., 2009). For such a purpose, we have transformed musical scores into numerical matrices relative to couples of parameters.

In order to obtain our 4 x 4 matrices, we have firstly divided each musical score into segments. We have then chosen a couple of parameters, e.g., pitch and time of start (onset) and have evaluated the mean values of all values of pitch and of all values of onset in the segment. We have then evaluated the distance (normalised between 0 and 1) of the parameters (a value of pitch and a value of onset) associated with each note in the examined segment with respect to the corresponding mean value (mean pitch and mean onset). Counting the number of notes with a pitch value in a given horizontal segment of the matrix with respect to the mean value of pitches, and counting an onset value in a given vertical segment of the matrix with respect to the mean value of onsets, we have constructed the matrix pitch-onset for the segment considered. We have followed the same procedure for all segments, and for the various couples of parameters.

We have then evaluated the trace distance between matrices relative to the same couple of parameters. The rate of variation of the trace distance between matrices gives information about the memory in open quantum systems, where matrices represent different quantum states. Since we have normalised musical matrices on the number of notes of each segment, the evaluation of the rate σ is simply the

[8] For a musical application of times series to music and detailed references, see Mannone (2013).

[9] Category theory is a branch of mathematics, embodied in a diagrammatic framework, that easily allows the description of transformations of transformations, and the creation of nested structures. A *category* is constituted by objects (visually represented as points), and morphisms between them (arrows), whose composition is associative and has the identity element. A category can be described as an object in itself; mappings between categories are *functors*.

evaluation of differences between matrices corresponding to pairs of parameters for each musical segment. More details are provided in the Appendix. Positive values of σ represent increasing distances.

The fundamental difference between measuring memory in physics and measuring it in music is that in physics the higher the distance between initial equal states, the higher the memory; whereas in music the higher the distance, the lower the musical similarity between segments and thus the lower the musical memory.

To check the consistency of our quantifier, $\mathcal{M}_C$, that is, the measured amount of memory, we have applied this method to two simple cases: two equal bars in the first case (Fig 1), two very different bars in the second (Fig. 2). We have found that, in these simple cases, there is an agreement between the information given by the quantifier and the memory as intuited by listening.

We have applied our method to evaluate the amount of memory of three compositions, that, by ear, show a different quantity of memory. These compositions are the vocal part of the aria *Dolente Immagine* by Bellini (Bellini, 1935), the first piece from the oboe suite *Solo* by Maderna (Maderna, 1971), and *Metamorphosis III* by Glass (Glass, 1988). Qualitatively, the amount of memory of the works by Bellini (one with the common structure A-B-A') and Glass (an example of minimalism) is higher than that of Maderna's score (an example of the avant-garde). The values obtained of $\mathcal{M}_C$ correspond to these expectations. That is, $\mathcal{M}_C$ has a higher value in scores to which one, by ear, attributes a larger amount of memory.

The proposed method is a further tool that can be utilised by musicians and musicologists to connect mathematical and physical concepts to musical concepts. It could, for example, be used to see if different composers in various historical periods can be characterised by a given mean memory level of their compositions, and also to compare two similar pieces (Cherlin, 1998; Mazzola et al., 2011; Kramer, 1992). An average value of musical memory could, for example, characterise the works of the same composer, or those of composers in the same artistic movement, or indeed those in a given musical form (fugue, theme with variations, fantasia, etc.). The degree of musical non-Markovianity can help identify the productions of specific composers, comparing a particular piece with the rest of their musical output. This technique could be automated in each step, to allow easy comparisons between larger musical compositions. Machine learning would constitute an invaluable tool for such an endeavour. An intermediate stage of machine learning to extract lines automatically and divide them into segments would be required to measure memory in more complex and extended musical works. In fact, this would make easier the application of our method to, for example, a complete symphony, allowing us to find motifs permeating the entire piece. Our algorithm can be applied to larger systems; everything we need is .txt files with numerical parameters extracted from the musical composition, and their subdivision into segments.

Information retrieved through musical non-Markovianity should in the future be compared with systematic listener studies and music cognition data. In fact, we expect a correlation between the amount of memory caught by the musical non-Markovianity quantifier and the qualitative judgement of musical memory as assessed by listeners. While musically educated people could provide a more refined judgement, we would expect a substantial degree of agreement between the human-assessed mean amount of musical memory and the software-assessed degree of musical memory. Such a comparison could constitute the object of a new experimental study. A listening study could also help us improve our method. To build up our matrices, we consider for each parameter a centre and distances from it. To make a distinction between up and down, we can use negative values for items below the centre. The difference in information between the computation with all positive or with all negative values can be compared against listeners' assessments of musical memory. For example, we could check if segments such as the one shown in Figure 2 lead to similar judgements, irrespective of the order. This could be helpful to assess the amount of information loss while using positive-only rather than positive and negative distances.

The idea of non-Markovian dynamics, as it is used in physics, can thus lead to a useful parameter to characterise musical structures and identify individual musical pieces. This idea can have further developments in the domain of Music Information Retrieval (MIR) for musical genre recognition.

We have focused on Western musical pieces from different epochs, but it is in principle also possible to assess the degree of musical non-Markovianity of non-Western compositions, provided that a MIDI transcription, with approximate pitches, durations, and intensities, is available. In the case of improvised and non-notated compositions, a transcription could also allow the application of our method.

As future developments of our study, the idea of musical non-Markovianity can enrich the most recent research in quantum computing applied to music (Miranda, 2021). Quantum computing is now being used to compose generative music.[10] The concept of musical non-Markovianity might contribute to musical quantum computing as well. In particular, we could develop a new gate to shape the degree of memory in musical segments:

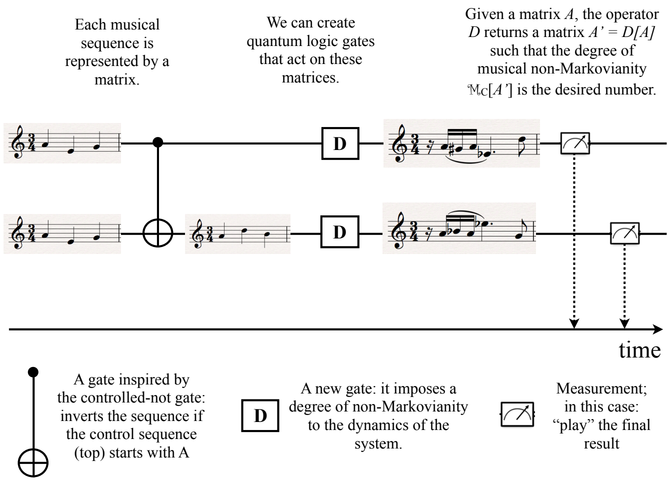

Figure 9: A possible new gate for a musical quantum circuit.

For example, given a musical segment s and a pair of parameters, we can evaluate the matrix A and its degree of musical non-Markovianity: $\mathcal{M}_C(A) = d$. To obtain a transformed segment s' (with matrix A') whose degree of non-Markovianity is the desired d', we can shape an operator D returning a matrix $A' = D(A)$ such that $\mathcal{M}_C(A') = d'$. The original musical segment s can thus be modified into a segment s', such that $s' \to A' \to \mathcal{M}_C(A') = d'$. For the original segment, we had $s \to A \to \mathcal{M}_C(A) = d$. This process can be formalised as a quantum gate D, taking as inputs A (associated with s) and d' and giving as output A' (associated with s'), whose degree of non-Markovianity is d'. In this way, we can build musical controllers inspired by quantum circuits, that is, with musical segments as "states" and quantum gates as operations on them, creating circuits like that presented in Fig. 9. Thus, we can imagine the development of "memory controllers" as time-evolution operators to be added to quantum circuits, regulating the number of repetitions of segments and the creation of internal loops according to their action on pitch, rhythm, harmony, or loudness "spaces".

Musical non-Markovianity can ultimately add a physical flavour to *Quantum Computer Music*, connecting musical beauty with hidden processes in nature.

8 Appendix

8.1 Building Musical Matrices

To obtain musical matrices, we have developed an algorithm with the following steps (the code for this, in C, can be found at: https://github.com/medusamedusa/musical_non-Markovianity):

[10] As an example of the process, the result of measurement of nine qubits subjected to the Hadamard gate is fed into an algorithm calculating triplets, which is then used to obtain parameters for sound synthesis (Miranda, 2020).

1. Evaluation of the normalised distance, for each coordinate, from its mean value.[11]
2. Counting the number of notes in a given interval (from the mean value), e.g., $\in [0,25]$, $[25,0.5]$, $[0.5,0.75]$, $[0.75,1]$.

We choose, as an example, only the couple of parameters frequency (ν) *versus* time of start (τ) of each note in the same segment. We consider the short musical segment of Fig. 10 as an example.

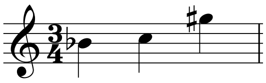

Figure 10: A simple musical segment to find the frequency-start matrix.

The musical parameters are frequencies, durations, starts, and intensities. In this example, for simplicity, there is no intensity indication. Let us consider only frequencies and starts. The frequencies (indicated as the semitone distances from middle C) are $\nu_1 = 10$, $\nu_2 = 12$, $\nu_3 = 20$, and the times of start are as follows (the durations are all equal, and correspond to a crotchet = 4 in our scale): $\tau_1 = 0$, $\tau_2 = 4$, $\tau_3 = 8$. The distances from the mean values are

$$\delta\nu_1 = 4,\ \delta\nu_2 = 2,\ \delta\nu_3 = 6;\ \ \delta\tau_1 = 4,\ \delta\tau_2 = 0,\ \delta\tau_3 = 4.$$

To normalise these distances, the algorithm finds minimum and maximum distance values:

$$\delta\nu_{min} = 2,\ \delta\nu_{max} = 6;\ \delta\tau_{min} = 0,\ \delta\tau_{max} = 4.$$

The normalised distances are evaluated as follows (if δ_{max} and δ_{min} are equal, the algorithm does not normalise):

$$\delta\nu_i^N = \frac{\delta\nu_i - \delta\nu_{min}}{\delta\nu_{max} - \delta\nu_{min}},\ \ \delta\tau_i^N = \frac{\delta\tau_i - \delta\tau_{min}}{\delta\tau_{max} - \delta\tau_{min}}$$

where $i = 1, 2, 3$ (in the example considered there are only three notes). In our example we obtain:

$$\delta\nu_1^N = 0.5,\ \delta\nu_2^N = 0,\ \delta\nu_3^N = 1;\quad \delta\tau_1^N = 1,\ \delta\tau_2^N = 0,\ \delta\tau_3^N = 1.$$

Now it is possible to construct the frequency-start matrix for this segment. The matrix will contain, in the rows, the number of notes with a normalised distance between 0 and 0.25, 0.25 and 0.5, 0.5 and 0.75, and 0.75 and 1 from the mean value of start; and, in the columns, the number of notes with a normalised distance between 0 and 0.25, 0.25 and 0.5, 0.5 and 0.75, and 0.75 and 1 from the mean value of frequency.

To avoid the difficulty of the different lengths of each segment (each segment can contain a different number of notes, the notes can have different durations, and the segment in total can have different durations, ...), we divide each matrix element by the total number of notes in the segment considered. In this way, we obtain a matrix, and the sum of each element is 1. Thus, we can easily compare segments of different lengths, and with different numbers of notes. In our simple example, the matrix is:

$$\rho = \begin{pmatrix} 0.33 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.33 \\ 0 & 0 & 0 & 0.33 \end{pmatrix}.$$

The note distribution into matrix slots depends upon the size of the matrix and the number of notes with similar characteristics. Notes with the same onsets and pitches are counted inside the same matrix elements. Once we choose the matrix size, the code computes its matrix elements. The longer and more complex the musical example, the more distributed the musical notes will be. We first build a matrix for each musical segment. Then, we compare the obtained matrices to measure eventual changes in the corresponding musical score.

[11] From a music-theoretic perspective, variations of a musical segment can go either way: higher or lower than the mean. For the purposes of our research, it is irrelevant if the added/modified notes are lower or higher than a given pitch, for example, as long as they are added, and thus the original segment is modified. In future research, a distinction between sides of the mean could be considered to find more fine-grained information.

8.2 Musical Scores

In this subsection, we present the annotated scores for each musical piece considered in Section 5, and we provide information concerning the comparison of segments.

In Bellini's piece, we compared the first segment of section A with the first segment of section B, the first segment of A with the first segment of A', the second segment of A with the second segment of A', and the third segment of A with the third segment of A'. Concerning Maderna's piece, it has not been possible to operate a univocal division of the piece into segments as for Bellini's composition. We thus decided to focus approximately on time duration and to halve the composition, with a first section containing the first five segments, and the second containing segments from 6 to 10, plus the 11th. We thus performed comparisons between segments 1 and 6, 2 and 7, 3 and 8, 4 and 9, 5 and 10, and 6 and 11. From the five-page Glass composition, we include the first page. It was possible to divide Glass's piece clearly into segments and to make cross-comparisons, as has been done for the Bellini.

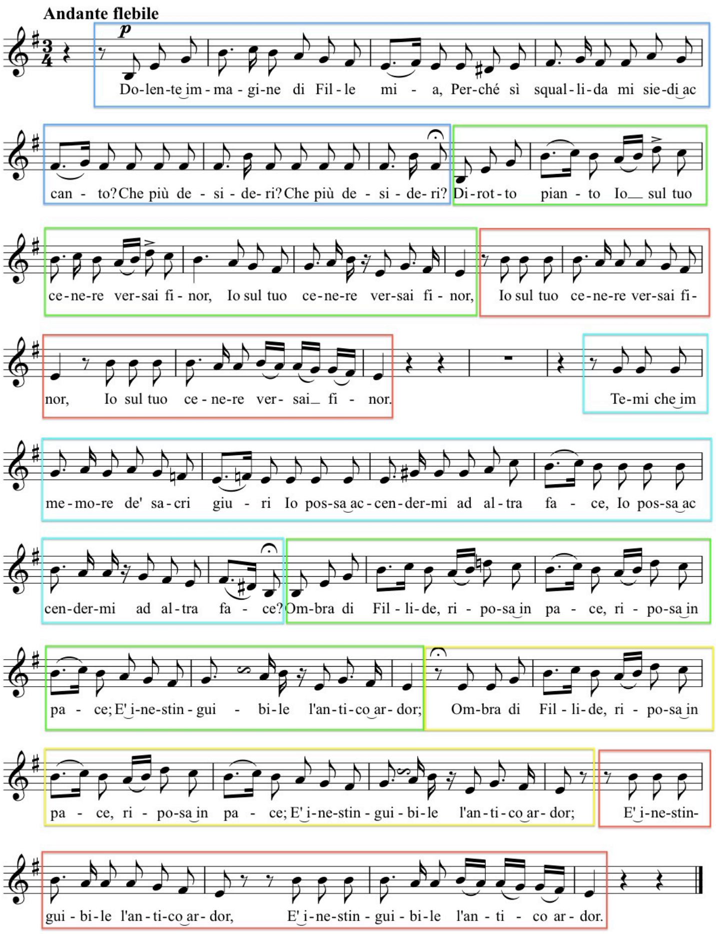

Figure 11: Annotated vocal part of *Dolente Immagine* by Vincenzo Bellini. Section A is composed of three segments (shown in blue, green and red); Section B of one segment (turquoise); and Section A' of three segments (green, yellow, red).

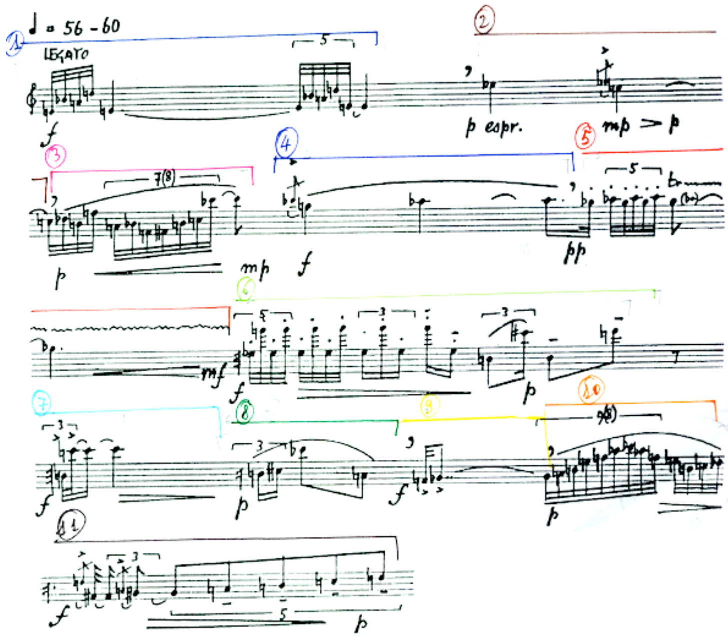

Figure 12: Annotated score of *Solo* by Bruno Maderna (from the composer's manuscript).

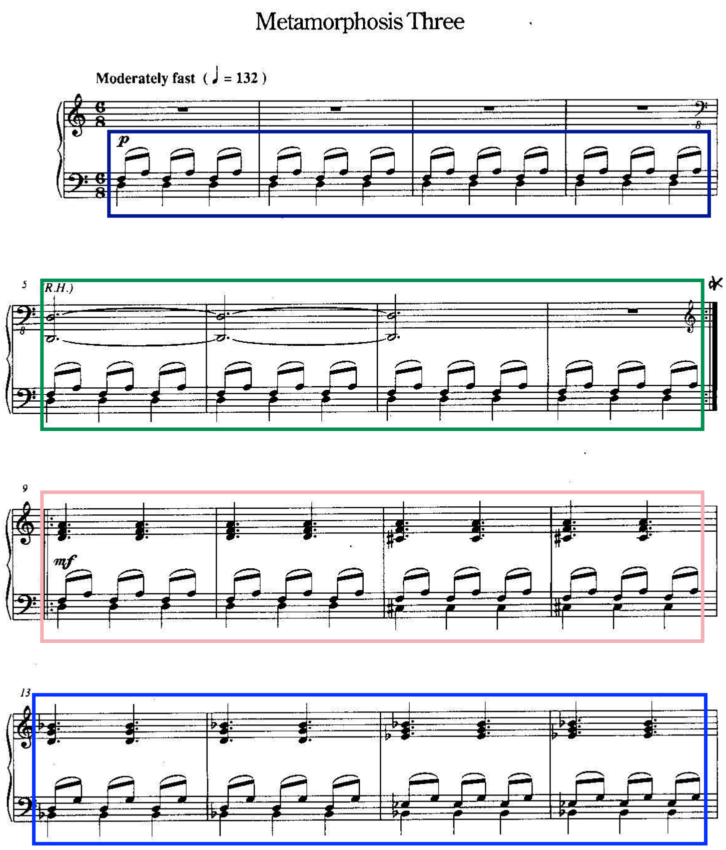

Figure 13: Annotated score of the first page of *Metamorphosis III* by Philip Glass.

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